ON STOCHASTIC GENERATORS OF COMPLETELY POSITIVE COCYCLES.

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Abstract. A characterisation of the generators of quantum stochastic cocycles of completely positive (CP) maps is given in terms of the complete dissipativity (CD) of its form-generator. The pseudo-Hilbert dilation of the stochastic form-generator and the pre-Hilbert dilation of the corresponding dissipator is found. The general form of the linear continuous structural maps for the algebra of all bounded operators is derived and the quantum stochastic flow for the corresponding cocycle is outlined. It is proved that any w*-analytical bounded CD form-generator give rise to a quantum stochastic CP cocycle over a von Neumann algebra.

1. Quantum Stochastic CP Cocycles and their Generators.

The quantum filtering theory [1] provides examples for a new type of irreversible quantum dynamics, described by one-parameter cocycles: $\phi = (\phi_t)_{t>0}$ of completely positive stochastic maps $\phi_t(\omega) : \mathcal{B} \to \mathcal{B}$ of an operator algebra $\mathcal{B} \subseteq \mathcal{B}(\mathfrak{h})$. The cocycle condition

$$\phi_s(\omega) \circ \phi_r(\omega^s) = \phi_{r+s}(\omega)$$

means the stationarity, with respect to the shift $\omega^s = \{\omega(t+s)\}$ of a given stochastic process $\omega = \{\omega(t)\}$. Such maps are in general unbounded, but normalized, $\phi_t(I) = M_t$ to an operator-valued martingale $M_t = \epsilon_t[M_s] \geq 0$ with $M_0 = 1$, or a positive submartingale: $M_t \geq \epsilon_t[M_s]$, for all $s > t$, where ϵ_t is the conditional expectation with respect to the history up to time t.

In the most general case, the stochastically differentiable family ϕ with respect to a quantum stationary process, with independent increments $A^s(t) = A(t+s) - A(s)$ generated by a finite dimensional Itô algebra is described by the quantum stochastic equation

$$\mathrm{d}\phi_t(X) = \phi_t \circ \alpha^\mu_\nu(X)\,\mathrm{d}A^\nu_\mu := \sum_{\mu,\nu} \phi_t(\alpha^\mu_\nu(X))\,\mathrm{d}A^\nu_\mu, \qquad X \in \mathcal{B} \tag{1}$$

with the initial condition $\phi_0(X) = X$, for all $X \in \mathcal{B}$. Here $A^\nu_\mu(t)$ with $\mu \in \{-, 1, ..., d\}$, $\nu \in \{+, 1, ..., d\}$ are the standard time $A^+_-(t) = tI$, annihilation $A^m_-(t)$, creation $A^+_n(t)$ and exchange $A^m_n(t)$ operator integrators with $m, n \in \{1, ..., d\}$. The infinitesimal increments $\mathrm{d}A^\mu_\nu(t) = A^{t\mu}_s(\mathrm{d}t)$ are formally defined

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by the Hudson-Parthasarathy multiplication table [2] and the $\flat$-property [3],

$$\mathrm{d}A^{\beta}_{\mu}\mathrm{d}A^{\nu}_{\gamma} = \delta^{\beta}_{\gamma}\mathrm{d}A^{\nu}_{\mu}, \qquad A^{\flat} = A, \qquad (2)$$

where δ^{β}_{γ} is the usual Kronecker delta over the indices $\beta \in \{-, 1, ..., d\}$, $\gamma \in \{+, 1, ..., d\}$and $A^{\flat\mu}_{-\nu} = A^{\nu\dagger}_{-\mu}$ with respect to the reflection $-(-) = +$, $-(+) = -$ of the indices $(-, +)$ only, such that $-m = m$, $-n = n$. The structural maps $\alpha^{\mu}_{\nu} : \mathcal{B} \to \mathcal{B}$ for the $*$-cocycles $\phi^{*}_{t} = \phi_t$, where $\phi^{*}_{t}(X) = \phi_t\left(X^{\dagger}\right)^{\dagger}$ should obviously satisfy the $\flat$-property $\alpha^{\flat} = \alpha$, where $\alpha^{\flat\nu}_{-\mu} = \alpha^{\mu *}_{-\nu}$, $\alpha^{\mu *}_{\nu}(X) = \alpha^{\mu}_{\nu}\left(X^{\dagger}\right)^{\dagger}$, even in the case of nonlinear ϕ_t and so α^{μ}_{ν}. If the coefficients α^{μ}_{ν} are independent of t, ϕ satisfies the cocycle property $\phi_s \circ \phi^{s}_{r} = \phi_{s+r}$, where ϕ^{s}_{t} is the solution to (1) with $A^{\mu}_{\nu}(t)$ replaced by $A^{s\mu}_{\nu}(t)$.

Let us prove that the "spatial" part $\boldsymbol{\lambda} = (\lambda^{\mu}_{\nu})^{\mu\neq+}_{\nu\neq-}$ of the tensor $\lambda = \alpha + \delta$ for a CP cocycle ϕ with $\delta^{\mu}_{\nu}(X) = X\delta^{\mu}_{\nu}$ must be conditionally CP in the following sense.

Theorem 1. *Suppose that the quantum stochastic equation (1) with $\phi_0(X) = X$ has a completely positive solution in the sense of positive definiteness of the matrix $[\phi_t(X_{kl})]$, $\forall t > 0$ given by an arbitrary positive definite matrix $[X_{kl}]$ with the elements $X_{kl} \in \mathcal{B}$. Then the matrix $\boldsymbol{\lambda}$ of structural maps $\lambda^{\mu}_{\nu}(X) = \alpha^{\mu}_{\nu}(X) + \delta^{\mu}_{\nu}(X)$ is conditionally completely positive,*

$$\sum_{k,l}\langle \boldsymbol{\eta}_k | \boldsymbol{\iota}(X_{kl})\boldsymbol{\eta}_l\rangle = 0 \Longrightarrow \sum_{k,l}\langle \boldsymbol{\eta}_k | \boldsymbol{\lambda}(X_{kl})\boldsymbol{\eta}_l\rangle \geq 0$$

where $[X_{kl}] \geq 0, \boldsymbol{\eta} \in \mathfrak{h} \otimes \mathbb{C}^{d+1}$ with respect to the degenerate representation $\boldsymbol{\iota} = (\iota^{\mu}_{\nu})^{\mu\neq+}_{\nu\neq-}$, $\iota^{\mu}_{\nu}(X) = X\delta^{+}_{\nu}\delta^{\mu}_{-}$, both written in the matrix form as

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda & \lambda_{\bullet} \\ \lambda^{\bullet} & \lambda^{\bullet}_{\bullet} \end{pmatrix}, \qquad \boldsymbol{\iota}(X) = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \qquad (3)$$

with $\lambda = \alpha^{-}_{+}$, $\lambda^{m} = \alpha^{m}_{+}$, $\lambda_{n} = \alpha^{-}_{n}$, $\lambda^{m}_{n} = \delta^{m}_{n} + \alpha^{m}_{n}$,where $\delta^{m}_{n}(X) = X\delta^{m}_{n}$, such that

$$\lambda\left(X^{\dagger}\right) = \lambda(X)^{\dagger}, \qquad \lambda^{n}\left(X^{\dagger}\right) = \lambda_{n}(X)^{\dagger}, \qquad \lambda^{m}_{n}\left(X^{\dagger}\right) = \lambda^{n}_{m}(X)^{\dagger}. (4)$$

Proof. Let us denote by $\mathcal{D}$ the $\mathfrak{h}$-span $\left\{\sum_f \xi^f \otimes f^{\otimes} \middle| \xi^f \in \mathfrak{h}, f^{\bullet} \in \mathbb{C}^d \otimes L^2(\mathbb{R}_+)\right\}$ of coherent (exponential) functions $f^{\otimes}(\tau) = \bigotimes_{t\in\tau} f^{\bullet}(t)$, given for each finite subset $\tau = \{t_1, ..., t_n\} \subseteq \mathbb{R}_+$ by tensor products $f^{n_1,...,n_N}(\tau) = f^{n_1}(t_1)...f^{n_N}(t_N)$, where $f^n, n = 1, ..., d$ are square-integrable complex functions on $\mathbb{R}_+$ and $\xi^f = 0$ for almost all $f^{\bullet} = (f^n)$. The co-isometric shift T_s intertwining $A^s(t)$ with $A(t) = T_s A^s(t) T^{\dagger}_{s}$ is defined on $\mathcal{D}$ by $T_s(\eta \otimes f^{\otimes})(\tau) = \eta \otimes f^{\otimes}(\tau + s)$. The complete positivity of the quantum stochastic adapted map ϕ_t into the $\mathcal{D}$-forms $\langle\chi|\,\phi_t(X)\psi\rangle$, for $\chi, \psi \in \mathcal{D}$ can be obviously written as

$$\sum_{k,l}\sum_{f,h}\left\langle \xi^{f}_{k}\middle|\, \phi_t(f^{\bullet}, X_{kl}, h^{\bullet})\,\xi^{h}_{l}\right\rangle \geq 0, \qquad (5)$$

for any operator-matrix $[X_{kl}] \geq 0$, where

$$\langle\eta|\,\phi_t(f^{\bullet}, X, h^{\bullet})\,\eta\rangle = \left\langle \eta \otimes f^{\otimes}\middle|\, \phi_t(X)\,\eta \otimes h^{\otimes}\right\rangle e^{-\int_t^{\infty} f^{\bullet}(s)^{*}h^{\bullet}(s)\mathrm{d}s},$$

$\xi^f \neq 0$ only for a finite subset of $f^{\bullet} \in \{f^{\bullet}_{i}, i = 1, 2, ...\}$. If the $\mathcal{D}$-form $\phi_t(X)$ satisfies the stochastic equation (1), the $\mathfrak{h}$-form $\phi_t(f^{\bullet}, X, h^{\bullet})$ satisfies the differential

equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi_t\left(f^\bullet,X,h^\bullet\right) &= f^\bullet\left(t\right)^* h^\bullet\left(t\right)\phi_t\left(f^\bullet,X,h^\bullet\right)+\phi_t\left(f^\bullet,\alpha_+^-\left(X\right),h^\bullet\right)\\ &+\sum_{m=1}^d f^m\left(t\right)^*\phi_t\left(f^\bullet,\alpha_+^m\left(X\right),h^\bullet\right)+\sum_{n=1}^d h^n\left(t\right)\phi_t\left(f^\bullet,\alpha_n^-\left(X\right),h^\bullet\right)\\ &+\sum_{m,n=1}^d f^m\left(t\right)^* h^n\left(t\right)\phi_t\left(f^\bullet,\alpha_n^m\left(X\right),h^\bullet\right)\end{aligned}$$

where $f^\bullet\left(t\right)^* h^\bullet\left(t\right)=\sum_{n=1}^d f^n\left(t\right)^* h^n\left(t\right)$. The positive definiteness, (5), ensures the conditional positivity

$$\sum_{k,l}\sum_{f,h}\langle\xi_k^f|X_{kl}\xi_l^h\rangle=0\Rightarrow\sum_{k,l}\sum_{f,h}\left\langle\xi_k^f\right|\lambda_t\left(f^\bullet,X_{kl},h^\bullet\right)\xi_l^h\right\rangle\geq 0$$

of the form $\lambda_t\left(f^\bullet,X,h^\bullet\right)=\frac{1}{t}\left(\phi_t\left(f^\bullet,X,h^\bullet\right)-X\right)$ for each $t>0$ and any $[X_{kl}]\geq 0$. This applies also for the limit λ_0 at $t\downarrow 0$, coinciding with the quadratic form

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\phi_t\left(f^\bullet,X,h^\bullet\right)\right|_{t=0}=\sum_{m,n}\bar a^m\lambda_n^m\left(X\right)c^n+\sum_m\bar a^m\lambda^m\left(X\right)+\sum_n\lambda_n\left(X\right)c^n+\lambda\left(X\right),$$

where $a^\bullet=f^\bullet\left(0\right),\quad c^\bullet=h^\bullet\left(0\right)$, and the λ's are defined in (4). Hence the form

$$\sum_{k,l}\sum_{\mu,\nu}\langle\eta_k^\mu|\,\lambda_\nu^\mu\left(X_{kl}\right)\eta_l^\nu\rangle:=\sum_{k,l}\langle\eta_k|\,\lambda\left(X_{kl}\right)|\eta_l\rangle$$

$$+\sum_{k,l}\left(\sum_n\langle\eta_k|\,\lambda_n\left(X_{kl}\right)\eta_l^n\rangle+\sum_m\langle\eta_k^m|\,\lambda^m\left(X_{kl}\right)\eta_l\rangle+\sum_{m,n}\langle\eta_k^m|\,\lambda_n^m\left(X_{kl}\right)\eta_l^n\rangle\right)$$

with $\eta=\sum_f\xi^f,\quad\eta^\bullet=\sum_f\xi^f\otimes a^\bullet$, where $a^\bullet=f^\bullet\left(0\right)$ is positive if $\sum_{k,l}\langle\eta_k|X_{kl}\eta_l\rangle=0$ for a positive-definite $[X_{kl}]$. The components η and $\eta^\bullet$ of these vectors are independent because for any $\eta\in\mathfrak{h}$ and $\eta^\bullet=\left(\eta^1,...,\eta^d\right)\in\mathfrak{h}\otimes\mathbb{C}^d$ there exists such a function $a^\bullet\longmapsto\xi^a$ on $\mathbb{C}^d$ with a finite support, that $\sum_a\xi^a=\eta,\quad\sum_a\xi^a\otimes a^\bullet=\eta^\bullet$, namely, $\xi^a=0$ for all $a^\bullet\in\mathbb{C}^d$ except $a^\bullet=0$, for which $\xi^a=\eta-\sum_{n=1}^d\eta^n$ and $a^\bullet=e_n^\bullet$, the n-th basis element in $\mathbb{C}^d$, for which $\xi^a=\eta^n$. This proves the complete positivity of the matrix form $\boldsymbol{\lambda}$, with respect to the matrix representation $\boldsymbol{\iota}$ defined in (4) on the ket-vectors $\boldsymbol{\eta}=(\eta^\mu)$. ∎

2. A Dilation Theorem for the Form-Generator.

The conditional complete positivity of the structural map $\boldsymbol{\lambda}$ with respect to the degenerate representation $\boldsymbol{\iota}$ written in the matrix form (4) obviously implies the positivity of the dissipation form

$$\sum_{X,Z}\langle\boldsymbol{\eta}_X|\,\boldsymbol{\Delta}\left(X,Z\right)\boldsymbol{\eta}_Z\rangle:=\sum_{k,l}\sum_{\mu,\nu}\langle\eta_k^\mu|\,\Delta_\nu^\mu\left(X_k,X_l\right)\eta_l^\nu\rangle\,,\qquad(6)$$

where $\eta^-=\eta=\eta^+$ and $\eta_k=\eta_{X_k}$ for any (finite) sequence $X_k\in\mathcal{B}$, $k=1,2,...$, corresponding to non-zero$\boldsymbol{\eta}_X\in\mathfrak{h}\otimes\mathbb{C}^{d+1}$. Here $\boldsymbol{\Delta}=(\Delta_\nu^\mu)_{\nu\neq-}^{\mu\neq+}$ is the stochastic

dissipator, given by the blocks

$$\begin{aligned}
\Delta_n^m(X,Z) &= \alpha_n^m\left(X^\dagger Z\right)+X^\dagger Z\delta_n^m,\\
\Delta_n^-(X,Z) &= \alpha_n^-\left(X^\dagger Z\right)-X^\dagger\alpha_n^-(Z)=\Delta_+^n(Z,X)^\dagger\\
\Delta_+^-(X,Z) &= \alpha_+^-\left(X^\dagger Z\right)-X^\dagger\alpha_+^-(Z)-\alpha_+^-\left(X^\dagger\right)Z+X^\dagger DZ,
\end{aligned}$$

where $D=\lambda(I)\le 0$ ($D=0$ for the case of the martingale M_t). In the linear case this means that the matrix-valued map $\lambda_\bullet^\bullet=[\lambda_n^m]$, is completely positive, and as follows from the next theorem, at least for the algebra $\mathcal{B}=\mathcal{B}(\mathfrak{h})$ the maps λ, λ^m, λ_n have the following form

$$\begin{aligned}
\lambda^m(X) &= \varphi^m(X)-K_m^\dagger X, \qquad \lambda_n(X)=\varphi_n(X)-XK_n \qquad (7)\\
\lambda(X) &= \varphi(X)-K^\dagger X-XK, \qquad \varphi(I)\le K+K^\dagger
\end{aligned}$$

where $\boldsymbol{\varphi}=(\varphi_\nu^\mu)_{\nu\neq-}^{\mu\neq+}$ is a completely positive bounded map from $\mathcal{B}$ into the matrices of operators with the elements $\varphi_n^m=\lambda_n^m,\varphi_+^m=\varphi^m,\varphi_n^-=\varphi_n,\varphi_+^-=\varphi:\mathcal{B}\to\mathcal{B}$.

In order to make the formulation of the dilation theorem as concise as possible, we need the notion of the $\flat$-representation [1] of the unital $*$-multiplicative structure of the algebra $\mathcal{B}$ in a pseudo-Hilbert space $\mathcal{E}=\mathfrak{h}\oplus\mathcal{K}\oplus\mathfrak{h}$ with respect to the indefinite metric

$$(\xi|\,\xi)=2\operatorname{Re}\left(\xi^-\middle|\,\xi^+\right)+\left\|\xi^\circ\right\|^2+\left\|\xi^+\right\|_D^2$$

for the triples $\xi=(\xi^\mu)^{\mu=-,\circ,+}\in\mathcal{E}$, where $\xi^-,\xi^+\in\mathfrak{h}$, $\quad\xi^\circ\in\mathcal{K}$, $\quad\mathcal{K}$ is a pre-Hilbert space, and $\|\eta\|_D^2=\langle\eta|\,D\eta\rangle$. Define the $(d+2)\times(d+2)$ matrix $\alpha=[\alpha_\nu^\mu]$ also for $\mu=+$ and $\nu=-$, by

$$\alpha_\nu^+(X)=0=\alpha_-^\mu(X),\qquad\forall X\in\mathcal{B},$$

and then one can extend the summation in (1) so it is also over $\mu=+$, and $\nu=-$. By such an extension the multiplication table for $\mathrm{d}A(\alpha)=\alpha_\nu^\mu\mathrm{d}A_\mu^\nu$ can be written as

$$\mathrm{d}A(\beta)\,\mathrm{d}A(\gamma)=\mathrm{d}A(\beta\gamma)$$

in terms of the usual matrix product $(\beta\gamma)_\nu^\mu=\beta_\alpha^\mu\gamma_\nu^\alpha$ and the involution $\alpha\mapsto\alpha^\flat$ can be obtained by the pseudo-Hermitian conjugation $\alpha_\beta^{\flat\nu}=G^{\nu\gamma}\alpha_\gamma^{\mu*}G_{\mu\beta}$ respectively to the indefinite metric tensor $G=[G_{\mu\nu}]$ and its inverse $G^{-1}=[G^{\mu\nu}]$, given by

$$G=\begin{bmatrix}0&0&I\\0&I_\circ^\circ&0\\I&0&D\end{bmatrix},\qquad G^{-1}=\begin{bmatrix}-D&0&I\\0&I_\circ^\circ&0\\I&0&0\end{bmatrix}$$

with an arbitrary D, where $I_\circ^\circ$ is the identity operator in $\mathcal{K}$, being equal $I_\bullet^\bullet=[I\delta_n^m]_{n=1,\dots,d}^{m=1,\dots,d}$ in the case of $\mathcal{K}=\mathfrak{h}\otimes\mathbb{C}^d$.

Theorem 2. *The following are equivalent:*

(1) *The dissipation form (6), defined by the $\flat$-map α with $\alpha_+^-(I)=D$, is positive definite:* $\sum_{X,Z}\langle\boldsymbol{\eta}_X|\,\boldsymbol{\Delta}(X,Z)\,\boldsymbol{\eta}_Z\rangle\ge 0$.

(2) *There exists a pre-Hilbert space $\mathcal{K}$, a unital $*$- representation j in $\mathcal{B}(\mathcal{K})$,*

$$j\left(X^\dagger Z\right)=j(X)^\dagger j(Z),\quad j(I)=I,$$

of the multiplication structure of $\mathcal{B}$, a (j,i)-derivation of $\mathcal{B}$ with $i(X)=X$,

$$k\left(X^\dagger Z\right)=j(X)^\dagger k(Z)+k\left(X^\dagger\right)Z,$$

having values in the operators $\mathfrak{h} \to \mathcal{K}$*, the adjoint map* $k^*(X) = k\left(X^\dagger\right)^\dagger$*, with the property*

$$k^*\left(X^\dagger Z\right) = X^\dagger k^*(Z) + k^*\left(X^\dagger\right) j(Z)$$

of (i,j)*-derivation in the operators* $\mathcal{K} \to \mathfrak{h}$*, and a map* $l : \mathcal{B} \to \mathcal{B}$ *having*

$$l\left(X^\dagger Z\right) = X^\dagger l(Z) + l\left(X^\dagger\right) Z + k^*\left(X^\dagger\right) k(Z),$$

with the adjoint $l^*(X) = l(X) + [D, X]$*, such that* $\lambda(X) = l(X) + DX$*,*

$$\lambda_n\left(X^\dagger\right) = k(X)^\dagger L_n^\circ + X^\dagger L_n^- = \lambda^n(X)^\dagger,$$

and $\lambda_n^m(X) = L_m^{\circ\dagger} j(X) L_n^\circ$ *for some operators* $L_n^\circ : \mathfrak{h} \to \mathcal{K}$ *having the adjoints* $L_n^{\circ\dagger}$ *on* $\mathcal{K}$ *and* $L_n^- \in \mathcal{B}$*.*

(3) *There exists a pseudo-Hilbert space,* $\mathcal{E}$*, namely,* $\mathfrak{h}\oplus\mathcal{K}\oplus\mathfrak{h}$ *with the indefinite metric tensor* $G = [G_{\mu\nu}]$ *given above for* $\mu, \nu = -, \circ, +$*, and* $D = \lambda(I)$*, a unital* $\flat$*-representation* $\jmath = [\jmath_\nu^\mu]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$ *of the multiplication structure of* $\mathcal{B}$ *on* $\mathcal{E}$ *:*

$$\jmath\left(X^\dagger Z\right) = \jmath(X)^\flat \jmath(Z), \quad \jmath(I) = I$$

with $\jmath(X)^\flat = G^{-1}\jmath(X)^\dagger G$*, given by*

$$\jmath_\circ^\circ = j, \quad \jmath_+^\circ = k, \quad \jmath_\circ^- = k^*, \quad \jmath_+^- = l, \quad \jmath_-^- = i = \jmath_+^+$$

and all other $\jmath_\nu^\mu = 0$*, and a linear operator* $\mathbf{L} : \mathfrak{h}\oplus\mathfrak{h}^\bullet \to \mathcal{E}$*, where* $\mathfrak{h}^\bullet = \mathfrak{h}\otimes\mathbb{C}^d$*, with the components* $(L^\mu, L_\bullet^\mu)$*,*

$$L^- = 0, \quad L^\circ = 0, \quad L^+ = I, \quad L_\bullet^- = \left(L_n^-\right), \quad L_\bullet^\circ = \left(L_n^\circ\right), \quad L_\bullet^+ = 0,$$

and $\mathbf{L}^\flat = \begin{pmatrix} I & 0 & D \\ 0 & L_\circ^\bullet & L_+^\bullet \end{pmatrix} = \mathbf{L}^\dagger G$*, where* $L_\circ^\bullet = L_\bullet^{\circ\dagger}, L_+^\bullet = L_\bullet^{-\dagger}$ *, such that*

$$\mathbf{L}^\flat \jmath(X) \mathbf{L} = \boldsymbol{\lambda}(X), \qquad \forall X \in \mathcal{B}.$$

(4) *The structural map* $\boldsymbol{\lambda} = \boldsymbol{\alpha} + \boldsymbol{\delta}$ *is conditionally positive-definite with respect to the matrix representation* $\boldsymbol{\iota}$ *in* (4) *:*

$$\sum_X \boldsymbol{\iota}(X)\boldsymbol{\eta}_X = 0 \Longrightarrow \sum_{X,Z} \langle \boldsymbol{\eta}_X | \boldsymbol{\lambda}\left(X^\dagger Z\right) \boldsymbol{\eta}_Z\rangle \geq 0.$$

Proof. Similar to the dilation theorem in [4], see also [5]. ∎

3. The structure of the w*-analytical CP cocycles with bounded generators.

The structure (7) of the linear form- generator for CP cocycles over $\mathcal{B} = \mathcal{B}(\mathfrak{h})$ is a consequence of the well known fact that the linear derivations k, k^* of the algebra $\mathcal{B}(\mathfrak{h})$ of all bounded operators on a Hilbert space $\mathfrak{h}$ are spatial, $k(X) = j(X)L - LX, \quad k^*(X) = L^\dagger j(X) - XL^\dagger$, and so

$$l(X) = \frac{1}{2}\left(L^\dagger k(X) + k^*(X) L + [X, D]\right) + i[H, X],$$

where $H^\dagger = H$ is a Hermitian operator in $\mathfrak{h}$. The structural map $\boldsymbol{\lambda}$ whose components are composed (as in (7)) into the sums of the components φ_ν^μ of a CP map $\boldsymbol{\varphi} : \mathcal{B} \to \mathcal{B}\otimes\mathcal{M}\left(\mathbb{C}^{d+1}\right)$ and left and right multiplications, are obviously conditionally completely positive with respect to the representation $\boldsymbol{\iota}$ in (4). As follows from the dilation theorem in this case, there exists a family

$L_- = L = L_+$, $L_n = L_n^\circ$, $n = 1, ..., d$ of linear operators $L_\nu : \mathfrak{h} \to \mathcal{K}$, having adjoints $L_\mu^\dagger : \mathcal{K} \to \mathfrak{h}$ such that $\varphi_\nu^\mu (X) = L_\mu^\dagger j(X) L_\nu$, .

The next theorem proves that these structural conditions which are sufficient for complete positivity of the cocycles, given by the equation (1), are also necessary if the structural map $\boldsymbol{\lambda}$ is w*-analytic [6] and bounded on the unit ball of a von-Neumann algebra $\mathcal{B}$. Thus the equation (1) for a completely positive w*-analytical cocycle with bounded stochastic derivatives has the following general form

$$\mathrm{d}\phi_t (X) + \phi_t \left(K^\dagger X + XK - L^\dagger j(X) L\right) \mathrm{d}t = \sum_{m,n=1}^{d} \phi_t \left(L_m^* j(X) L_n - X\delta_n^m\right) \mathrm{d}A_m^n$$

$$+ \sum_{m=1}^{d} \phi_t \left(L_m^* j(X) L - K_m^* X\right) \mathrm{d}A_m^+ + \sum_{n=1}^{d} \phi_t \left(L^* j(X) L_n - XK_n\right) \mathrm{d}A_-^n, \qquad (8)$$

given by a w*-analytical representation j. This gives a quantum stochastic generalization of the branching norm-continuous semigroups with the nonlinear generators [6] . If the space $\mathcal{K}$ can be embedded into the direct sum $\mathfrak{h} \otimes \mathbb{C}^d = \mathfrak{h} \oplus ... \oplus \mathfrak{h}$ of d copies of the initial Hilbert space $\mathfrak{h}$ such that $j(X) = [X\delta_n^m]$, this equation corresponds to the Lindblad form [7] for the generator $\lambda = \alpha_+^-$. But in the contrast to the Lindblad equation, it can be resolved in the form $\phi_t(X) = F_t^\dagger X F_t$, where $F = (F_t)_{t>0}$ is an (unbounded) cocycle in the tensor product $\mathfrak{h} \otimes \mathcal{F}$ with Fock space $\mathcal{F}$ over the Hilbert space $\mathbb{C}^d \otimes L^2(\mathbb{R}_+)$ of the quantum noise of dimensionallity d. The cocycle F satisfies the quantum stochastic equation

$$\mathrm{d}F_t + KF_t \mathrm{d}t = \sum_{i,n=1}^{d} \left(L_n^i - I\delta_n^i\right) F_t \mathrm{d}A_i^n + \sum_{i=1}^{d} L^i F_t \mathrm{d}A_i^+ - \sum_{n=1}^{d} K_n F_t \mathrm{d}A_-^n, \qquad (9)$$

where L_n^i and L^i are the operators in $\mathfrak{h}$, defining

$$\begin{aligned} \varphi_n^m (X) &= \sum_{i=1}^{d} L_m^{i\dagger} X L_n^i, \qquad \varphi(X) = \sum_{i=1}^{d} L^{i\dagger} X L^i \\ \varphi^m (X) &= \sum_{i=1}^{d} L_m^{i\dagger} X L^i, \qquad \varphi_n (X) = \sum_{i=1}^{d} L^{i\dagger} X L_n^i \qquad (10) \end{aligned}$$

with $\sum_{i=1}^d L^{i\dagger} L^i = K + K^\dagger$if M_t is a martingale ($\leq K + K^\dagger$if submartingale) .

Theorem 3. *Let the structural maps $\boldsymbol{\lambda}$ of the quantum stochastic cocycle ϕ over a von-Neumann algebra $\mathcal{B}$ be w*-analytic and bounded:*

$$\|\lambda\| < \infty, \qquad \|\lambda_\bullet\| = \left(\sum_{n=1}^{d} \|\lambda_n\|^2\right)^{\frac{1}{2}} = \|\lambda^\bullet\| < \infty, \qquad \|\lambda_\bullet^\bullet\| = \|\lambda_\bullet^\bullet (I)\| < \infty,$$

where $\|\lambda\| = \sup\{\|\lambda(X)\| : \|X\| < 1\}$, $\|\lambda_\bullet^\bullet (I)\| = \sup\{\langle \eta^\bullet, \lambda_\bullet^\bullet (I) \eta^\bullet \rangle \,|\, \|\eta^\bullet\| < 1\}$, *and ϕ_t is a CP cocycle, satisfying equation (1) with $\phi_0(X) = X$. Then they have the form (7) written as*

$$\boldsymbol{\lambda}(X) = \boldsymbol{\varphi}(X) - \boldsymbol{\iota}(X) \boldsymbol{K} - \boldsymbol{K}^\dagger \boldsymbol{\iota}(X)$$

with $\varphi = \varphi_+^-$, $\varphi^m = \varphi_+^m$, $\varphi_n = \varphi_n^-$ *and* $\varphi_n^m = \lambda_n^m$, *composing a* w^**-analytical bounded CP map.*

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi & \varphi_\bullet \\ \varphi^\bullet & \varphi_\bullet^\bullet \end{pmatrix}, \quad \text{and} \quad \boldsymbol{K} = \begin{pmatrix} K & K_\bullet \\ K^\bullet & K_\bullet^\bullet \end{pmatrix}$$

with arbitrary $K^\bullet, K_\bullet^\bullet$. *The equation (8) has the unique CP solution, given by the iteration of the quantum stochastic integral equation*

$$\phi_t(X) = V_t^\dagger X V_t + \int_0^t V_s^\dagger \beta_\nu^\mu\left(\phi_{t-s}^s(X)\right) V_s \mathrm{d}A_\mu^\nu(s)$$

where $\beta_\nu^\mu(X) = \varphi_\nu^\mu(X) - X\delta_\nu^\mu$ *and* V_t *is the vector cocycle* $V_r^s V_s = V_{r+s}$, *resolving the quantum stochastic differential equation*

$$\mathrm{d}V_t + KV_t\mathrm{d}t + \sum_{m=1}^d K_m V_t \mathrm{d}A_-^m = 0$$

with the initial condition $V_0 = I$ *in* $\mathfrak{h}$ *and with* $V_r^s = T_r^\dagger V_r T_s$, *shifted by the time-shift co-isometry* T_s *in* $\mathcal{D}$.

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